Errata for the Third Edition of
Handbook of Differential Equations

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NOTES:

1. The latest errata is available from `http://www.mathtable.com/zwillinger/errata/`.
2. The home page for this book is `http://www.mathtable.com/hode/`.
3. You can reach the author at `ZwillingerBooks@gmail.com`.

1. Section 6, **Classification of Partial Differential Equations**, page 35.

   (a) The equation between equations (6.3) and (6.4) currently has the line:

   $$u_{xy} = u_{\eta\eta}\eta_x\eta_y + \boxed{2}u_{\eta\zeta}(\eta_x\zeta_y + \eta_y\zeta_x) + u_{\zeta\zeta}\zeta_x\zeta_y + u_\eta\eta_{xy} + u_\zeta\zeta_{xy},$$

   which is incorrect, it should have been:

   $$u_{xy} = u_{\eta\eta}\eta_x\eta_y + u_{\eta\zeta}(\eta_x\zeta_y + \eta_y\zeta_x) + u_{\zeta\zeta}\zeta_x\zeta_y + u_\eta\eta_{xy} + u_\zeta\zeta_{xy},$$

   (b) The equation after equation (6.4) currently has the line:

   $$\overline{B} = A\zeta_x\eta_x + B(\zeta_x\eta_y + \zeta_y\eta_x) + 2C\zeta_y\eta_y,$$

   which is incorrect, it should have been (a “2” was missing)

   $$\overline{B} = 2A\zeta_x\eta_x + B(\zeta_x\eta_y + \zeta_y\eta_x) + 2C\zeta_y\eta_y,$$

   (Thanks to Hans Weertman for these corrections.)

2. Section 6, **Classification of Partial Differential Equations**, page 38.

   Add a new **Example 4**

   Consider the 3-by-3 system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, where A is constant. The eigenvalues of A satisfy $\lambda^3 + P\lambda^2 + Q\lambda + R = 0$ where

   $$\begin{aligned} P &= -\operatorname{trace}(A) \\ Q &= \frac{1}{2}\left(P^2 - \operatorname{trace} A^2\right) \\ R &= -\det A \end{aligned}$$

The surface S, defined by $27R^2 + \left(4P^3 - 18PQ\right)R + \left(4Q^3 - P^2Q^2\right) = 0$, divides the real solutions from the complex solutions. When $\{P, Q, R\}$ are real, the surface S can be split up into two surfaces S_{1a} and S_{1b} given by $R = R_a$ and $R = R_b$ where:

$$R_a = \frac{1}{3}P\left(Q - \frac{2}{9}P^2\right) - \frac{2}{27}\left(P^2 - 3Q\right)^{3/2}$$
$$R_b = \frac{1}{3}P\left(Q - \frac{2}{9}P^2\right) + \frac{2}{27}\left(P^2 - 3Q\right)^{3/2}$$

The classification of solution trajectories in (P, Q, R) space is:

(a) node / node / node
  i. nodes are stable if: $[P > 0,\ 0 < Q < \frac{P^2}{3},\ 0 < R < R_b]$
  ii. nodes are unstable if: $[P < 0,\ 0 < Q < \frac{P^2}{3},\ R_a < R < 0]$

(b) node / node / star node
  i. nodes are stable if: $[P > 0,\ \frac{P^2}{4} < Q < \frac{P^2}{3},\ R = R_a]$ or $[P > 0,\ 0 < Q < \frac{P^2}{3},\ R = R_b]$
  ii. nodes are unstable if: $[P < 0,\ \frac{P^2}{4} < Q < \frac{P^2}{3},\ R = R_a]$ or $[P < 0,\ 0 < Q < \frac{P^2}{3},\ R = R_b]$

(c) star node / star node / star node
  i. nodes are stable if: $[P > 0,\ Q = \frac{P^2}{3},\ R = R_a = R_b = \frac{P^3}{27}]$
  ii. nodes are unstable if: $[P < 0,\ Q = \frac{P^2}{3},\ R = R_a = R_b = \frac{P^3}{27}]$

(d) line node-saddle / line node-saddle / node
  i. nodes are stable if: $[P > 0,\ 0 < Q < \frac{P^2}{4},\ R = 0]$
  ii. nodes are unstable if: $[P < 0,\ 0 < Q < \frac{P^2}{4},\ R = 0]$

(e) line node-saddle / line node-saddle / star node
  i. nodes are stable if: $[P > 0,\ Q = \frac{P^2}{4},\ R = R_a = 0]$
  ii. nodes are unstable if: $[P < 0,\ Q = \frac{P^2}{4},\ R = R_b = 0]$

(f) node / saddle / saddle
  i. node is stable if: $[P \geq 0,\ Q < \frac{P^2}{4},\ R_a < R < 0]$ or $[P < 0,\ Q < 0,\ R_a < R < 0]$
  ii. node is unstable if: $[P \leq 0,\ Q < \frac{P^2}{4},\ 0 < R < R_a]$ or $[P > 0,\ Q < 0,\ 0 < R < R_b]$

(g) star node / saddle / saddle
  i. node is stable if: $[P \geq 0,\ Q < \frac{P^2}{4},\ R = R_a]$ or $[P < 0,\ Q < 0,\ R = R_a]$
  ii. node is unstable if: $[P \leq 0,\ Q < \frac{P^2}{4},\ R = R_b]$ or $[P > 0,\ Q < 0,\ R = R_b]$

(h) line node-saddle / line node-saddle / no flow
  i. both line node-saddles are stable if: $[P > 0,\ Q = 0,\ R = 0]$
  ii. both line node-saddles are unstable if: $[P < 0,\ Q = 0,\ R = 0]$
  iii. one line node-saddle is stable and the other unstable if: [for all P, $Q < 0,\ R = 0$]

(i) focus / stretching

i. focus is stable if: $[P \geq 0,\ Q > \frac{P^2}{4},\ R < 0]$ or $[P \geq 0,\ Q < \frac{P^2}{4},\ R < R_a]$ or $[P < 0,\ Q > 0,\ R < PQ]$ or $[P < 0,\ Q > 0,\ R < R_a]$

ii. focus is unstable if: $[P < 0,\ Q > \frac{P^2}{3},\ PQ < R < 0]$ or $[P < 0,\ \frac{P^2}{4} < Q < \frac{P^2}{3},\ R_b < R < 0]$ or $[P < 0,\ 0 < Q < \frac{P^2}{3},\ PQ < R < R_a]$

(j) focus / compressing

i. focus is stable if: $[P > 0,\ Q > \frac{P^2}{4},\ 0 < R < PQ]$ or $[P > 0,\ \frac{P^2}{4} < Q < \frac{P^2}{3},\ 0 < R < R_a]$ or $[P > 0,\ 0 < Q < \frac{P^2}{3},\ R_b < R < PQ]$

ii. focus is unstable if: $[P \leq 0,\ Q > \frac{P^2}{4},\ R > 0]$ or $[P \leq 0,\ Q < \frac{P^2}{4},\ R < R_b]$ or $[P > 0,\ Q > 0,\ R < PQ]$ or $[P > 0,\ Q > 0,\ R < R_a]$

(k) focus / no flow

i. focus is stable if: $[P > 0,\ Q > \frac{P^2}{4},\ R = 0]$

ii. focus is unstable if: $[P < 0,\ Q > \frac{P^2}{4},\ R = 0]$

(l) center

i. stretching if $[P < 0,\ Q > 0,\ R = PQ]$

ii. compressing if $[P > 0,\ Q > 0,\ R = PQ]$

iii. no flow if $[P = 0,\ Q > 0,\ R = 0]$

Reference: Chong M. S., Perry A. E., & Cantwell B. J. (1990). A general classification of three-dimensional flow fields. Physics of Fluids, A 2(5), pages 765–777.

3. Section 7, **Compatible Systems**, page 41, Special Case 3. The text for this special case is incorrect. It should be replaced with:

   In the special case of $r = 1$, we have a system of m equations in m dependent variables. These equations do not require any side conditions.

   (Thanks to Rusty Humphrey for this correction.)

4. Section 11, **Fixed Point Existence Theorems**, page 54

   (a) The name "Schrauder" should be "Schauder"

   (b) The following reference should be added:

   J. Schauder, "Der Fixpunktsatz in Funktionalräumen," **Studia Math.**, 2, (1930), 171–180.

   (Thanks to G. Friesecke for these corrections.)

5. Section 13, **Integrability of Systems**, page 65, Note number 11 contains "the sine–Gordan equation" when it should have "the sine–Gordon equation".

   (Thanks to Alain Moussiaux for this correction.)

6. Section 17, **Natural Boundary Conditions for a PDE**, page 77, The equation at the top of page 77, before equation (17.1) is now

$$J[\phi + h] - J[\phi] = \iint\limits_R \left\{ L_{\phi_t} h_t + L_{\phi_{x_j}} h_{x_j} + L_\phi \right\} dt\, d\mathbf{x} + O(||h||^2),$$

This is incorrect, it should have been (notice the last "h" was missing)

$$J[\phi + h] - J[\phi] = \iint\limits_R \left\{ L_{\phi_t} h_t + L_{\phi_{x_j}} h_{x_j} + L_\phi h \right\} dt\, d\mathbf{x} + O(||h||^2),$$

(Thanks to Zhuo Li for this correction.)

7. Section 27, **Canonical Forms**, page 118, reference number 2 is now

   Bateman, H. *Partial Differential Equations of Mathematical Physics*, Dover Publications, New York, 1944.

   Which is incorrect. The reference should have been

   Bateman, H. *Differential Equations*, Longmans, Green and Co., New York, 1926, pages 75–79.

   (Thanks to Ali Nejadmalayeri for this correction.)

8. Section 36, **Transformations of Second Order Linear ODEs – 1**, page 137, equation (36.6) is now $I(x) = \left(b - \frac{1}{4}a^2 - \frac{1}{2}\frac{da}{dx}\right)$, which is incorrect. It should be $I(x) = \left(b + \frac{1}{4}a^2 - \frac{1}{2}\frac{da}{dx}\right)$, (Thanks to Richard D. Rabbitt for this correction.)

9. Section 44.1.2, **Look-Up Technique**, page 169, the two equations

   (a) Painlevé–Ince – modified

   (b) Pinney

   are both missing the "$= 0$" that should at the end of each.

   (Thanks to Alain Moussiaux for these corrections.)

10. Section 44.1.3, **Look-Up Technique**, page 172, last equation before section 44.2, presently has

    $y^{(m)} = a\boxed{xy^{-m/2}}$

    This is incorrect, it should have been

    $y^{(m)} = a\boxed{yx^{-m/2}}$

    (Thanks to Flavio Noca for this correction.)

11. Section 50, **Clairaut's Equation**, page 216, the equation between (50.5) and (50.6) is now

$$y''[2(xy' - 2)x - 2y'] = 0$$

which is incorrect. This expression should be

$$y''[2(xy' - y)x - 2y'] = 0$$

(Thanks to Bruno Muratori for this correction.)

12. Section 53, **Contact Transformation**, page 227, the second equation in equation 53.7 has the form

$$\cdots = \left(2X^3 - 3X\right)^{1/3}$$

which is incorrect. This expression should be

$$\cdots = \boxed{C}\left(2X^3 - 3X\right)^{1/3}$$

(Thanks to Alain Moussiaux for this correction.)

13. Section 72, **Green's functions**, page 292, From above equation (72.9) to that equation the text is presently:

> Using the second method, we find the eigenvalues and eigenfunctions to be
>
> $$\lambda_n = \boxed{\frac{n\pi}{L}}, \qquad \phi_n(x) = \sin\lambda_n x = \sin\left(\frac{n\pi x}{L}\right),$$
>
> so that
>
> $$G(x;z) = \boxed{\frac{2L}{n\pi}}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi z}{L}\right).$$

which is incorrect; the text should have been

> Using the second method, we find the eigenvalues and eigenfunctions to be
>
> $$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \phi_n(x) = \sin\lambda_n x = \sin\left(\frac{n\pi x}{L}\right),$$
>
> so that
>
> $$G(x;z) = \sum_{n=1}^{\infty}\left(-\frac{2L^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi z}{L}\right).$$

(Thanks to Luis Alberto Fernandez for this observation.)

14. Section 79, **Integrating Functions**, page 326, note number 10, the following should be added:

> The general solution to $u_x = yu_y$ is $u = f(x + \log y)$, where f is an arbitrary function.

(Thanks to Alain Moussiaux for this observation.)

15. Section 80, **Interchanging Dependent and Independent Variables**, page 327,

    (a) In Example 3, the nonlinear equation is given as "$y''(x-y)y'^3$", which is incorrect. It should have been "$y''(y-x)y'^3$".
    (Thanks to Alain Moussiaux for this correction.)

    (b) In Note number 2, the reference to Bender and Orszag should be section 1.5, not 1.6.
    (Thanks to James Dare for this correction.)

    (c) A better citation for reference number 3 is: McAllister, B. L. and Thorne, C.J. "Reverse differential equations and others that can be solved exactly", *Studies Appl. Math*, 6, 1952.
    (Thanks to Daniele Ritelli for this correction.)

16. Section 85, **Reduction of order**, page 354, note number 2 presently contains

    More generally, if $\{z_1(x), \dots, z_p(x)\}$ are linearly independent solutions of equation (85.6), then the substitution

$$y(x) = \begin{bmatrix} z_1 & \dots & z_p & v \\ z_1' & \dots & z_p' & v' \\ \vdots & & \vdots & \vdots \\ z_1^{(p)} & \dots & z_p^{(p)} & v^{(p)} \end{bmatrix}$$

    reduces equation (85.7) to a linear ordinary differential equation of order $n-p$ for $v(x)$.

    This should be changed to

    More generally, if $\{z_1(x), \dots, z_p(x)\}$ are linearly independent solutions of equation (85.6), then the substitution

$$y(x) = \begin{bmatrix} z_1 & \dots & z_p & z \\ z_1' & \dots & z_p' & z' \\ \vdots & & \vdots & \vdots \\ z_1^{(p)} & \dots & z_p^{(p)} & z^{(p)} \end{bmatrix} \phi(x) \tag{1}$$

    where $\phi(x)$ need not be specified, reduces equation (85.6) to a linear ordinary differential equation of order $n-p$ for $y(x)$. The following explains why.

With the above, $y(x)$ can be written in the form

$$y(x) = A(x)z^{(p)} + B(x)z^{(p-1)} + \dots, \qquad A(x) \neq 0$$

and its derivatives have the form

$$y'(x) = A(x)z^{(p+1)} + \dots, \qquad y''(x) = A(x)z^{(p+2)} + \dots,$$

These equations can be used to eliminate $\{z^{(p)}, \dots, z^{(n)}\}$ and (85.6) will take the form

$$b_0 y^{(n-p)} + \dots + b_{n-p} y + V = 0 \tag{2}$$

where V is linear in the $\{z, z', \dots, z^{(p-1)}\}$.
We argue that $V \equiv 0$ as follows: Consider equation (2) as a differential equation of degree $p-1$ in z (via the V term). If $z = z_i$ (for any $i = 1, 2, \dots, p$) then $y = 0$ from equation (1). Hence, from equation (2) it must be that $V|_{z=z_i} = 0$. Hence $\{z_i\}_{i=1,2,\dots,p}$ is a collection of p linearly independent solutions to a differential equation of degree $p-1$; possible only if $V \equiv 0$.

(Thanks to Unal Goktas for this correction.)

17. Section 87, **Matrix Riccati Equations**, page 358. The second line in equation (87.4) is now

$$\frac{dy}{dt} = b(t)(y^2 - x^2) - 2a(t)xy \boxed{-} 2cy$$

Which is incorrect, it should have been

$$\frac{dy}{dt} = b(t)(y^2 - x^2) - 2a(t)xy + 2cy$$

(Thanks to both Peter Sherwood and Alain Moussiaux for this correction.)

18. Section 93, **Superposition**, page 373, the last line contains the equation

$$L[y] = y'' + a(x)y' + b(x) = f(x)$$

Which is incorrect. This should have been

$$L[y] = y'' + a(x)y' + b(x)\boxed{y} = f(x)$$

(Thanks to Young Kim for this correction.)

19. Section 95, **Variation of parameters**, page 380, the following should be added as note number 5:

 If the linear second order equation $L[y] = y'' + P_1(x)y' + P_0(x)y = R(x)$ has the homogeneous solutions $y_1(x)$ and $y_2(x)$ (i.e., $L[y_i] = 0$), then the solution to the original equation may be written as

$$\begin{aligned} y(x) &= -y_1(x)\int \frac{y_2(x)R(x)}{W(y_1,y_2)}\,dx + y_2(x)\int \frac{y_1(x)R(x)}{W(y_1,y_2)}\,dx, \\ &= y_1(x)\int \frac{\begin{vmatrix} 0 & y_2 \\ R & y_2' \end{vmatrix}}{W(y_1,y_2)}\,dx + y_2(x)\int \frac{\begin{vmatrix} y_1 & 0 \\ y_1'' & R \end{vmatrix}}{W(y_1,y_2)}\,dx \end{aligned}$$

 where $W(y_1,y_2) = y_1y_2' - y_1'y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian.

 If the linear third order equation $L[y] = y''' + P_2(x)y'' + P_1(x)y' + P_0(x)y = R(x)$ has the homogeneous solutions $y_1(x)$, $y_2(x)$, and $y_3(x)$ (i.e., $L[y_i] = 0$), then the solution to the original equation may be written as

$$\begin{aligned} y(x) = y_1(x)\int \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ R & y_2'' & y_3'' \end{vmatrix}}{W(y_1,y_2,y_3)}\,dx &+ y_2(x)\int \frac{\begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & R & y_3'' \end{vmatrix}}{W(y_1,y_2,y_3)}\,dx \\ &+ y_3(x)\int \frac{\begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & R \end{vmatrix}}{W(y_1,y_2,y_3)}\,dx \end{aligned} \tag{3}$$

 where $W(y_1,y_2,y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$ is the Wronskian.

20. Section 96, **Vector Ordinary Differential Equations** pages 384-385, In note number 9 the second equation is incorrect. All the text after "Alternately, if the . . . " should be deleted.

 (Thanks to Frankie Liu for this correction.)

21. Section 106, **Inverse Scattering**, page 416, the **Applicable to** statement should have at the end

 having the form of (106.2)

 (Thanks to G. Friesecke for this observation.)

22. Section 106, **Inverse Scattering**, page 418, Note number 5 gives a Lax pair for the equation $u_t + u_{xx} - 2uu_x = 0$, which is not quite the Burger's equation. (Notice the minus sign before the last term.)

 (Thanks to Bruno Muratori for this correction.)

23. Section 118, **Chaplygin's Method**, page 465, equations (118.5) and (118.6) and the surrounding text are now

    Then define $u_1(x)$ to be the solution of

$$y' = M(x)y + N(x), \qquad y(x_0) = y_0. \tag{118.5}$$

    and define $v_1(x)$ to be the solution of

$$y' = \widehat{M}(x)y + \widehat{N}(x), \qquad y(x_0) = y_0. \tag{118.6}$$

    Which is incorrect. This should have been (note that the definitions have been switched):

    Then define $v_1(x)$ to be the solution of

$$y' = M(x)y + N(x), \qquad y(x_0) = y_0. \tag{118.5}$$

    and define $u_1(x)$ to be the solution of

$$y' = \widehat{M}(x)y + \widehat{N}(x), \qquad y(x_0) = y_0. \tag{118.6}$$

    (Thanks to Bruno Van der Bossche for these corrections.)

24. Section 123, **Graphical Analysis: The Phase Plane**, pages 479, 480.

    (a) In the text for example 1 it says

        ... The curve figure 123.2 is given by determinant= $(\text{trace})^2$; only centers can occur along this curve.

        which is incorrect; it should have said

        ... The curve in figure 123.2 is given by determinant= $(\text{trace}/2)^2$. Centers occur along the curve defined by trace= 0.

    (b)

    (Thanks to Zhuo Li for these corrections.)

25. Section 136, **Monge's Method**, pages 523–524,

    (a) Equation (136.5) contains, in part

    $$\cdots = \boxed{\frac{\partial z}{\partial y}} + 6y$$

    which is incorrect. This expression should be

    $$\cdots = \boxed{\frac{\partial z}{\partial x}} + 6y$$

    (b) Equation (136.10) contains, in part

    $$\cdots + \psi\left(2\boxed{z} + y^2\right)$$

    which is incorrect. This expression should be

    $$\cdots + \psi\left(2\boxed{x} + y^2\right)$$

    (Thanks to Alain Moussiaux for this correction.)

26. Section 139, **Perturbation Method: Method of Averaging**, pages 532–533,

    (a) In equations (139.3) and (139.5) the last "cos" in each case should be a "sin".

    (b) The two equations in (139.9) are each missing a final closing parenthesis.

    (Thanks to Gerald Teschl for these corrections.)

27. Section 143, **Perturbation Method: Regular Perturbation**, page 554, equations (143.5 b) and (143.7 b) both have "$y_1(0) = 1$" which is incorrect; they should have been "$y_1(0) = 0$".

    (Thanks to Frank Scharf for this corrections.)

28. Section 145, **Picard Iteration**, page 561, note number one, the following should be added:

    > However, the successive approximations are guaranteed to converge to the true solution for all x sufficiently close to zero provided f is a continuously differentiable function.

    (Thanks to G. Friesecke for this observation.)

29. Section 148, **Soliton-Type Solutions**, pages 567–569,

    (a) In equation (148.3) the term cv_ζ should be $-cv_\zeta$.

    (b) In equation (148.4) the term $(v_\zeta)^2$ should be $\frac{1}{2}(v_\zeta)^2$.

    (c) An additional note should be added on page 569 to state

    With the standard choice of $A = B = 0$, the solution to (148.4) can be solved in terms of elementary functions:

    $$v(x) = \frac{3c}{\sigma}\left(\operatorname{sech}\left(\frac{\sqrt{c}x}{2}\right)\right)^2$$

    (Thanks to G. Friesecke for these corrections.)

30. Section 180, **Runge–Kutta Methods**, pages 691, 696

(a) Equation (180.3) is missing some "h" terms. Presently there is:

$$\begin{aligned} k_1 &= f(x_0, y_0), \\ k_2 &= f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1), \\ k_3 &= f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2), \\ k_4 &= f(x_0 + h, y_0 + k_3). \end{aligned} \tag{4}$$

which is incorrect. It should have been:

$$\begin{aligned} k_1 &= f(x_0, y_0), \\ k_2 &= f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}\boxed{h}k_1), \\ k_3 &= f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}\boxed{h}k_2), \\ k_4 &= f(x_0 + h, y_0 + \boxed{h}k_3). \end{aligned} \tag{5}$$

(b) Note number 9 is incorrect and should be deleted.

31. Section 172, **Pseudospectral Method**, page 772, presently has:

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{1}{3h}(u_{k+1} - u_{k-1}) - \frac{1}{6h}(u_{k+2} - u_{k-2}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{1}{2h}(u_{k+1} - u_{k-1}) - \frac{1}{3h}(u_{k+2} - u_{k-2}) + \frac{1}{30h}(u_{k+3} - u_{k-3}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} = \sum_{j=1}^{\infty} \frac{2(-1)^{j+1}}{jh}(u_{k+j} - u_{k-j}).$$

Which are all incorrect. They should have been:

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{2}{3h}(u_{k+1} - u_{k-1}) - \frac{1}{12h}(u_{k+2} - u_{k-2}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{3}{4h}(u_{k+1} - u_{k-1}) - \frac{3}{20h}(u_{k+2} - u_{k-2}) + \frac{1}{60h}(u_{k+3} - u_{k-3}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{jh}(u_{k+j} - u_{k-j}).$$

(Thanks to Didier Clamond for these corrections.)